Extreme pluralism in mathematics and logic

UANALYTICON

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Two components of platonism

'Mathematical platonism' is usually taken to have two components.

1. The existence of mathematical objects;
2. The objectivity of mathematical questions, even when those questions are undecidable in our best theories.

'Objective' is quite obscure, but a natural interpretation is as the denial of extreme pluralism.

Most of this talk will be focused on views of mathematics that are anti-objectivist in this sense, that is, that are extremely pluralist, whatever their stance on issue 1.

I'll mainly consider two such views:

Fictionalism;

Conventionalism.

I'll also mention another: the multiverse view (*aka* plenitudinous platonism). And maybe also a variant of the multiverse view due to Hilary Putnam.

Extension of issue 2 to logic? (where issue 1 doesn't arise)

An apparent advantage of conventionalism is that it seems to more easily extend to the case of logic, i.e. to make sense of extreme pluralism there.

I see the attractions of thinking that, but am somewhat skeptical.

Getting clearer about that requires getting clearer about what conventionalism amounts to in the *mathematical* case.

Fictionalism

I'll start with the fictionalist view of my *Science Without Numbers*. This view rejects both components of standard platonism.

The official focus of the book was on the "objects" component of platonism. But much of the motivation was to avoid having to take seriously questions like "Is $aleph_{17}$ the true size of the continuum?":

Since mathematical entities are fictitious, all one can say is that it is true according to some fictions that extend what has currently been adopted and false according to others.

Internal objectivity

If "the fiction currently adopted" doesn't decide a question, we might say it lacks "internal objectivity".

There is a certain vagueness in "what has currently been adopted", and so, as to how much is "internally objective". (Is the Gödel sentence of Zermelo-Fraenkel set theory settled by what's been currently adopted?)

But it doesn't much matter, because from a fictionalist point of view, questions about internal objectivity are of limited interest.

External objectivity (1)

They're of limited interest in that they all depend on which theories happen to have been adopted.

Suppose the mathematical community comes to unqualifiedly accept that there are measurable cardinals (and suppose this consistent with the ZFC axioms).

Then it would be *true "in" (according to) their fiction* (despite there being other reasonable fictions in which it isn't true). So the question of measurable cardinals would have become internally objective.

But there is still a reasonable ("external") sense in which it wouldn't be objective: its coming out "true" depends on historical accidents.

External objectivity (2)

In a similar sense, even the axioms of ZF aren't all objective; e.g. it is easy to imagine having adopted a fiction according to which all sets are countable and so the Power Set axiom fails.

(Indeed, this altered fiction would still have sufficed to develop nearly all 19^{th} century mathematics.)

External objectivity (3)

There's nothing in fictionalism that has to cast doubt on the objectivity of first order logic.

And one way to take what I've said is that first order logic (or maybe some extension, like a modal logic or a logic with a finiteness quantifier) exhausts the external objectivity of math.

In that case, *no* theories that posit mathematical entities are "externally objective", since they all rely on fictions.

External objectivity (4)

But a fictionalist might well want to grant a special status to number theory or some part of it, due to applications, without giving such status to ZF set theory. (E.g. on the grounds that number theory directly reflects the logic of the cardinality quantifiers, or that it directly reflects claims of consistency.) Can we regard this as a kind of "external objectivity"?

Sure, the notion of objectivity isn't clear enough to fight over.

But bringing such utility considerations into what counts as "externally objective" seems guaranteed to make this notion too rather vague and a matter of degree.

Fictionalism about logic? (1)

In the other direction, one could conceivably regard even first order logic as a fiction, with alternatives to it as competing fictions.

Since first order logic doesn't posit objects, that can't be motivated in the same way, but perhaps in some other way.

The issue isn't whether there can be fictions, even mathematical fictions, according to which different logics are correct. There can be (e.g. smooth infinitesimal analysis).

But that no more casts doubt on the objectivity of logic than fictions with different physics cast doubt on the objectivity of physics. It shows only that there is a distinction between the logic posited by a fiction and genuine logic.

Fictionalism about math is basically the view that there is no mathematics outside fictions. Fictionalism about logic would be analogous, and seems much harder to believe.

Fictionalism about logic? (2)

One could be fictionalist about aspects of first order logic, such as the application of the law of excluded middle in “suspicious” contexts (e.g. vague questions, Liar sentences, etc.).

But it’s natural to think that we need some “core logic” that isn’t a fiction, in which the theory of fictions can be developed.

(There needn’t be agreement as to which parts of first order logic are “core” in this sense: there’s nothing here to suggest an *indubitable* core.)

Kreisel's dictum

Kreisel is often quoted as saying that the issue of mathematical objects is far less important than the issue of mathematical objectivity.

Despite the vagueness in 'objectivity', I'm inclined to agree.

Kreisel meant that it's important to maintain the objectivity of math, even if you think there are no mathematical objects.

Whereas my view is that even if you think there are mathematical objects, so that fictionalism is wrong, you should still think that there is no more objectivity to math than fictionalism allows.

Plenitudinous platonism (1)

Plenitudinous platonism (Balaguer, Hamkins, Clarke-Doane) disagrees with fictionalism on ontology but pretty much agrees with it as regards objectivity.

It's hard to state the view very precisely, but the rough idea is that for any logically consistent mathematical theory (e.g. ZFC + the size of the continuum is $\aleph_{27}$), there are universes in which it's true and universes in which it's false, ***with none of them in any way privileged*** (beyond our having chosen to work in them, or been led to by historical accident).

The boldfaced is meant to rule out a unique "real universe" of which the others are mere sub-universes or forcing extensions or models guaranteed by the completeness theorem.

(Though Hamkins often calls the universes models. His temptation to do so is a symptom of the difficulty of making clear sense of the boldfaced clause.)

Plenitudinous platonism (2)

Advocates of this view often claim that it yields mathematical objectivity. But all that can be objective is the nature of the pluriverse, which amounts essentially to the objectivity of logical consistency.

There are extended pluriverse views that contain universes with different logics, and hence that purport to cast doubt on even the objectivity of logical consistency.

But again, it is hard to see how a pluriverse view could be articulated without making use of some core logic, thereby privileging those universes that accept this logic.

Plenitudinous platonism (3)

And perhaps you need some core mathematics too to develop the pluriverse, e.g. some minimal set theory?

After all, it's natural to compare two universes in the pluriverse via functions from one to the other. What is the set theory in which these functions live? (It's this problem that makes set-theoretic pluralism much less obvious than geometric pluralism.)

If we do need a core math, this would seem to privilege those universes that accept that core math.

So maybe the pluriverse view doesn't achieve as much anti-objectivity as it claims? Maybe it's less anti-objectivist than fictionalism?

(Putnam's variant of the pluriverse does better on this score.)

Mathematical conventionalism vs. fictionalism

Mathematical conventionalism is a form of anti-objectivism that purports to differ both from fictionalism and from plenitudinous platonism.

It differs from fictionalism in taking mathematical objects to "exist by convention", and taking claims about such objects to be "true by convention". (It allows for a lot of internal objectivity, i.e. within a convention, but is anti-objectivist in denying much external objectivity.)

Why isn't this fictionalism by another name? After all, a fictionalist grants that sets exist *according to the fiction of ZFC*, so hasn't the conventionalist just substituted 'convention' for 'fiction', and 'exists by' / 'true by' for 'exists according to' / 'true according to'?

Rudeness

Some people regard it as rude to mathematicians to call mathematical theories fictions. Maybe ‘convention’ is perceived as less rude?

I admit to some rhetorical excess in comparing mathematical theories to novels, but it was always an explicit part of the fictionalist view that mathematics is quite different from literary fictions in its careful deductive practices and its utility in application to the physical world. The point of the “fictionalism” label is simply that literal truth isn’t the goal of mathematics.

Anyway, if the rudeness consideration seems compelling, then by all means call it ‘conventionalism’ instead of ‘fictionalism’, if there are no other differences between the views.

Staying within (1)

Another difference between mathematical fictions/conventions and literary fictions: we rarely “stay within” a literary fiction for long, whereas we do for a mathematical “fiction”.

(In everyday life we say that “within the fiction” Sherlock Holmes lived on Baker Street, but that in reality there was no one who lived there with the name or characteristics attributed to him. Whereas only in special contexts do we step outside the mathematical fiction or convention.)

Perhaps the term ‘convention’ is more appropriate for a fiction that one goes outside of only in special circumstances? Maybe.

Staying within (2)

But the important point is that fictionalists and conventionalists don’t disagree over this:

(i) both think that typical mathematical discourse proceeds “within the fiction or convention” (and with no need of reminders that one is within a fiction or convention); and

(ii) both think it appropriate to go outside one’s mathematical fictions/conventions in philosophical contexts, or in defusing debates about whether the continuum hypothesis is really true.

Explicit vs. implicit

Perhaps only truth by *explicit* convention is like fictionalism?

Modern conventionalists like Jared Warren (2020) rightly insist that any interesting conventionalism involves *implicit* conventions; these are to be understood as *practices* of a certain kind, and what's true by convention is what those practices serve to legitimate.

But here too I think there is no difference from how fictionalism has been understood: fictionalists always assumed that even arithmetic talk prior to its explicit codification by Dedekind and Peano is best construed as a fiction.

Application (1)

Perhaps the conventionalist and fictionalist differ as regards application?

SWN took the view that while it's perfectly fine for most purposes to formulate physical theories using fictions (indeed that this is a practical necessity), still *we should be committed to the possibility* of reformulating these theories without fictions.

Given this assumption, mathematical fictionalism is only defensible if such theories can be developed in a "nominalistic" way that posits no mathematical entities.

Someone who calls themself a conventionalist could conceivably take this hardline view, saying that physics is a matter of fact not convention; but I suspect that few if any self-described conventionalists take that stand.

Application (2)

But in fact, most people who call themselves mathematical fictionalists today aren't hardliners either: they say that not only is there nothing wrong with stating physical theories using fictions (as of course everyone agrees), there's also no problem if the appeal to fiction in physical theories isn't entirely eliminable.

There's a division here, between "easy road" nominalists who apparently see no point in trying to find formulations that *limit* the use of fictions, and those like my current self who do see the value and think that a great deal of progress can be made toward limiting the use of fiction, but are skeptical that in every physical theory the goal of doing entirely without the fiction can be achieved in any interesting way.

So, there are differences as to how much dispensability one hopes for or requires; but I don't see, in any of this, a fundamental difference between conventionalism and fictionalism.

Not taking ontology seriously

Maybe conventionalism is like fictionalism but with the added claim that ontological questions aren't to be taken seriously?

That added claim might be good advice, independent of conventionalism. But conventionalism is supposed to offer *a reason* not to take it seriously, the reason being, it's all just convention anyway. That's the analog of the fictionalist saying, don't worry about mathematical ontology because it's all just *fiction* anyway.

Someone who advises us not to take ontological questions seriously is likely to think that this fictionalist rendition of the view doesn't capture its spirit: the view was supposed to undermine the need to go fictionalist.

But if so, then the conventionalist rendition doesn't capture its spirit either: the conventionalist, like the fictionalist, would be taking ontological questions seriously in insisting that mathematical objects exist merely by convention.

Thin/lightweight objects

Some who think we shouldn’t take questions of mathematical ontology seriously like to say of those who do that they fail to recognize that the kind of entities that mathematics posits are “thin” or “lightweight”, so are no worry.

But without cashing out the metaphor, it’s unhelpful: why aren’t thin, lightweight blobs of platoplasm just as mysterious as fat and heavy ones?

Conventionalists might be thought of as cashing out the metaphor: lightweight existence is merely existence *according to convention*. But of course this is analogous to what a fictionalist could say: lightweight existence is merely existence *according to fiction*.

Again, it’s hard to see how conventionalism is anything other than fictionalism by another name.

Byproducts of convention (1)

Still, the version of mathematical conventionalism recently advocated by Jared Warren does appear to be genuinely different.

Warren says that mathematical entities genuinely exist, as a "byproduct" of our using mathematical language in the way that we do, and that their genuine existence is "fully explained by" our use of mathematical language.

As he says, 'byproduct' isn't to be understood causally, and the "full explanation" is not a causal one: our use of language didn't bring mathematical entities into being. (The view that it did would create obvious problems for the application of mathematics to early epochs of the universe, or to counterfactual situations where linguistic practices were very different.)

'Byproduct' must be understood in some other way. But what?

Byproducts of convention (2)

A natural thought might be that "x exists as a byproduct of our linguistic practices" just means something like "our linguistic practices involve the acceptance of rules that entail that x exists".

But fictionalists too think that the linguistic practices within math involve accepting such rules. Since Warren insists that his view isn't fictionalist, his "byproduct" claim must involve more than that.

So what *does* he mean by mathematical existence being a byproduct of linguistic use? In what way does our acceptance of the rules fully explain mathematical existence?

Top-down vs. Bottom-up truth

Warren's answer is in terms of truth. He distinguishes between top-down and bottom-up approaches to truth.

On a bottom-up approach, truth is explained in terms of reference, and I think his view is that you'd need to have '2' standing for a number that is not a byproduct of convention for '2+2=4' to be true on a bottom-up view.

But on a top-down approach, you can explain the truth of '2+2=4' directly by our practices of accepting it, and in this explanation we don't merely get that '2+2=4' is true *according to those practices*, we get that it is genuinely true. (And its explicitly existential consequences, like '∃x(x+x=4)', will for similar reasons be genuine truths.)

Byproducts of convention (3)

But I don't see how this delivers what he presumably wants. What he presumably wants isn't just the metalinguistic claims *that '2+2=4' and '∃x(x+x=4)' are truths of our language*, but the object-level claims *that 2+2=4* and *that ∃x(x+x=4)*.

Of course, we'd get the object level from the meta-linguistic on a deflationary understanding of 'true': on such an understanding, the claim that '∃x(x+x=4)' is a truth of our language is *equivalent* to the claim that ∃x(x+x=4).

But Warren's "top-down truth" (explained in terms of acceptance) is *not* deflationary in this sense.

Byproducts of convention (4)

The step from '2+2=4' and '∃x(x+x=4)' being truths of our language to 2+2=4 and ∃x(x+x=4) could also be justified on a bottom up approach to semantics. That's because on these approaches, the object level claims are *preconditions* of the sentences being true.

But as preconditions, there is no hope of non-circularly establishing them from the truth of the sentences; that's why Warren insisted on using the alternative top-down approach to semantics.

If you want to define 'true' as 'is licensed by our basic practices', who's to stop you? But calling it that doesn't guarantee that it disquotes, and if you think it does, that needs an argument. Which is what?

Byproducts of convention (5)

Indeed, conventionalists usually recognize that the inference from being licensed by our basic practices to *disquotational* truth can't be good in general: if the claim to the existence of an all-powerful creator were licensed by our practices, such a being wouldn't exist even as a "byproduct" of those practices.

So conventionalists tend to restrict the argument to claims about entities not part of the causal order (perhaps calling them "lightweight"). But why should the argument be valid in this restricted case when it isn't valid generally?

Byproducts of convention (6)

I think we can acknowledge that if '$\exists x(x+x=4)$' is licensed by our basic practices, and those practices are taken to exclude a fictionalist interpretation, then those practices give a kind of *default justification* to the belief that $\exists x(x+x=4)$.

But that's compatible with the justification being undercut by other considerations, e.g. whatever considerations might motivate fictionalism.

Moreover, even if one could somehow get to the conclusion that if '$\exists x(x+x=4)$' is licensed by our basic practices then it is *overall* justifiable for us to believe it, this is weaker than Warren wants: it doesn't rule out the belief being justified for us but disquotationally false.

Avoiding “byproducts”

Here’s a genuine alternative to conventionalism being fictionalism in disguise:

Instead of having to invent a special sense in which a mathematical universe is a “byproduct” of convention, we might think of all the possible such “byproducts” as there all along; convention (that is, linguistic practice) just serves to pick one out.

But this just seems to be plenitudinous platonism in disguise!

And I think that few who call themselves conventionalists would be happy with the idea that the role of conventions is merely to pick out preexisting objects.

Plenitudinous platonism in disguise better than fictionalism in disguise? (1)

In some ways the plenitudinous platonist view accords better with what conventionalists want than does the fictionalist view:

(1) Plenitudinous platonism doesn't take ontological worries seriously (in a more serious sense than fictionalism doesn't);

(2) Plenitudinous platonism is perhaps better placed than fictionalism with regard to applications: it doesn't seem *all that* surprising if our formulation of physical theory has to depend on the region of the multiverse we've been led to work in.

This needn't deprive of interest the project of showing how much can be done in a mathematical-universe-independent way, but may be thought to make it less pressing.

The fictionalist can say the analogous thing, but in pluralist lingo it sounds more natural.

Plenitudinous platonism in disguise better than fictionalism in disguise? (2)

But plenitudinous platonism seems no better than fictionalism as regards extension to logic.

(1) The PP'ist can easily speak of universes where smooth infinitesimal analysis holds and hence in which the logic is intuitionist; but a fictionalist can analogously speak of fictions according to which that holds.

(2) Just as the fictionalist seems to need a core logic to develop the theory of fictions, similarly the PP'ist needs a core logic to develop the theory of the pluriverse.

Conventionalism about logic?

Does conventionalism do better at extending to logic?

Quine argued that it did worse, in his regress argument against logical conventionalism. But this argument assumed that conventions had to be explicit.

Dummett gave a similar argument, but without making that assumption.

Dummett’s argument

Dummett’s argument is that a conventionalist can regard some logical truths as direct registers of our linguistic practices, but must regard others as logical consequences of these more basic ones, so that there is need for a non-conventional consequence relation.

Warren responded, correctly in my view, that the distinction between basic and derivative arises only in regards to our *theorizing about* the practice.

And maybe in our theorizing we can use a conventionally-adopted logic?

Incoherent practices

Nonetheless, there seems to be a clear distinction between coherent and incoherent practices.

Incoherent practices are ones that involve a commitment to the acceptance and rejection of the same thing (typically, to the acceptance of everything and the rejection of everything).

For instance, our practices seem to involve a commitment to (i) reasoning classically while (ii) accepting the naive rules of truth and (iii) rejecting claims such as that the earth is flat. But that's an incoherent set of practices. (Curry's paradox shows that in classical logic, the naive truth rules lead to any conclusion at all.)

Given this, one might well think that we need a minimal or core logic to determine what is incoherent. If so, how can the core logic be conventional?

A more basic worry about logical conventionalism

In any case, conventionalism about all of logic, if viewed as distinct from fictionalism and PP'ism, has the same problem as mathematical conventionalism:

- It introduces an inflationary top-down notion of truth, where the truth of "snow is either white or it isn't" is made true by our practices;
- But in concluding that it's a byproduct of our practices that snow is either white or it isn't, it conflates this inflationary notion with a deflationary one (on which truth is merely a logical device that by its nature disquotes).

Is there a sense in which logical facts are "lightweight" and our practices can establish "lightweight" but not "heavyweight" facts? Again, this would need to be shown.

Final slide

A *partial* conventionalism about logic (that accepts a non-conventionalist core) can avoid this worry. (I'm somewhat sympathetic to it, for reasons connected to paradoxes of truth and properties.)

Conventionalism about all of *mathematics* also avoids this worry, if it accepts a collapse into either fictionalism or plenitudinous platonism.

But I don't clearly see any such fallback position for conventionalism about *all of logic* to collapse into.

THE END

Putnam variant of multiverse

Instead of many mathematical universes, there's a single infinite mathematical universe, but no privileged relations on it: e.g. no privileged set-theoretic membership relation. Different choices of the relation are the analog of different universes.

Our acceptance of sentences involving '$\in$' constrains which of all the possible binary relations on this "intrinsically featureless" universe count as set-theoretic. But if we haven't decided whether to accept the continuum hypothesis or Suslin's hypothesis, there's no sense in which an interpretation that settles these one way is "intended" and one that settles it another way is "unintended". And even if we've settled on say the Power Set axiom as part of set theory, or something else that entails uncountable sets, it's only because we've done so that the existence of uncountable sets comes out true under intended interpretations.

Which such functions are posited depend on the chosen membership relation, so the analog of the multiverse (the totality of different membership relations) will look different on one choice of membership relation than on another. But that's what a plenitudinous platonist should want.